Solution to Problem 13) Applying the method of integration by parts, we find

$$
\begin{aligned}
I_n &= \int_0^{\pi/2} \sin^n(x)\,\mathrm{d}x = \int_0^{\pi/2} \sin(x)\sin^{n-1}(x)\,\mathrm{d}x \\
&= -\cos(x)\sin^{n-1}(x)\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2(x)\sin^{n-2}(x)\,\mathrm{d}x \\
&= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2}(x)\,\mathrm{d}x = (n-1)(I_{n-2} - I_n) \\
&\qquad \rightarrow \quad I_n = [(n-1)/n]I_{n-2}. \qquad (1)
\end{aligned}
$$

The area under $\sin^n(x)$ is a monotonically decreasing function of n, which approaches zero as $n \to \infty$. Equation (1) shows that the ratio I_{n-2}/I_n approaches 1 as $n \to \infty$. Considering that the sequence is monotonically decreasing, we conclude that the ratio I_{n-1}/I_n of adjacent members of the sequence must also approach 1 as $n \to \infty$.

If n is an even integer, continuation of the procedure that has led to Eq.(1) will eventually stop at $I_0 = \pi/2$, in which case,

$$I_n = I_{2m} = \frac{\pi(n-1)!!}{2(n!!)} = \frac{\pi(2m-1)!!}{2(2m)!!}. \qquad (2)$$

In contrast, if n happens to be an odd integer, we will have $I_1 = \int_0^{\pi/2} \sin x\,\mathrm{d}x = 1$, and, therefore,

$$I_n = I_{2m+1} = \frac{(n-1)!!}{n!!} = \frac{(2m)!!}{(2m+1)!!}. \qquad (3)$$

The Wallis product emerges from Eqs.(2) and (3) above, if we now set the limit of I_{2m+1}/I_{2m} equal to 1 when $m \to \infty$, that is,

$$1 = \lim_{m\to\infty} \frac{I_{2m+1}}{I_{2m}} = \lim_{m\to\infty} \frac{(2m)!!}{(2m+1)!!} \times \frac{2(2m)!!}{\pi(2m-1)!!} \quad \rightarrow \quad \prod_{m=1}^{\infty} \left(\frac{2m}{2m-1} \cdot \frac{2m}{2m+1}\right) = \frac{\pi}{2}. \qquad (4)$$
